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ANALOGY BETWEEN REFINED BEAM THEORIES AND THE BERNOULLI–EULER THEORY†

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Abstract—Some refinements of the classical Bernoulli–Euler theory of the bending of beams are shown to be completely analogous to the effects produced by sources of self-stress in the Bernoulli–Euler beam. This is true for deflections, bending moments and shearing forces. They determine rotations, stresses and strains according to the specific refined theory under consideration. Thus, by analogy, differences between the results of various refined theories, having been the object of discussions in recent literature, become accessible to a systematic classification. In an Appendix, this strategy of treating refined theories from the point of view of the classical one is put into the more general context of the Theory of Science.

INTRODUCTION

This paper is concerned with quasi-static engineering theories of plane, flexural deformations of straight, linear elastic beams loaded by lateral forces, leading to constitutive equations for bending moments M and shearing forces Q of the following rather general type:

$$M = -B(w_{,xx} - \beta(\psi + w_{,x})_{,x}), \tag{1}$$

$$Q = S(\psi + w_{,x}). \tag{2}$$

Linearized geometrical conditions are assumed to be valid, and $(.)_{,x}$ denotes the derivative with respect to the axial coordinate. β is a theory-dependent factor. B and S denote the bending and shearing stiffness, respectively. w is the transverse deflection (of the axis, or an averaged deformation), and ψ is a (generalized) cross-sectional rotation.

Without loss of substantial generality, but for the sake of comparison, the examples given in the last section of the paper will be restricted to those of homogeneous beams of constant, rectangular cross-section.

In that case $B = 2h^3E/3$ holds in all of the following theories, with Young's modulus E. The rectangle's height is $2h$, while its width is taken to be unity for convenience. In the classical Bernoulli–Euler theory (see Grüning (1914, p. 494) and the literature cited there) of beams rigid in shear, $S \to \infty$, there is the kinematical constraint $\psi + w_{,x} = 0$, and β does not need to be considered. On the contrary, in the Timoshenko theory of shear-deformable beams [Timoshenko (1921); see Grüning (1914, p. 508), von Karman (1910, p. 334) for some earlier contributions], we have $\beta = 1$. Frequently, $S = 5Eh/[6(1+\nu)]$ is to be found in the literature; ν denotes Poisson's ratio. Additional to Timoshenko's refinement of the Bernoulli–Euler theory, the refined theories of Levinson (1981), Rehfield and Murthy (1982) and of Rychter (1988) will be considered, where the isotropic versions of the latter two

† Transmitted by Franz Ziegler.

Table 1. Characteristic parameters β, S and β/s in non-dimensional form for isotropic beams of rectangular cross-section, according to various refined theories

	β	S/Eh	$\beta Eh/S$
Timoshenko (1921)	1	$5/(6+6\nu)$	$6(1+\nu)/5$
Levinson (1981)	$4/5$	$2/(3+3\nu)$	$6(1+\nu)/5$
Rehfield and Murthy (1982)			
(a)	$(8+5\nu)/(10+10\nu)$	$2/(3+3\nu)$	$3(8+5\nu)/20$
(b)	$(8+5\nu)/(8+9\nu)$	$20/(24+27\nu)$	$3(8+5\nu)/20$
(c)	$3(8+5\nu)/(20+25\nu)$	$4/(4+5\nu)$	$3(8+5\nu)/20$
Rychter (1988)	$(12+5\nu)/(10+10\nu)$	$1/(1+\nu)$	$(12+5\nu)/10$

theories are taken into account. Corresponding values of β and $\beta Eh/S$ for rectangular cross-sections are listed in Table 1. They have been re-calculated from the referenced papers. Note some remarkable coincidences between the various theories, which does not necessarily mean coincidences in the basic assumptions.

It is noted that eqns (1) and (2) are also valid in the case of sandwich beams. For sandwich beams with thin surface layers, where the bending stiffness of the facings, shearing stresses in the facings and longitudinal normal stresses in the core are neglected, we have $\beta = 1$, see e.g. Plantema (1966) for appropriate expressions for B and S. For thick-sandwich beams, an extension has been given recently by Gordaninejad and Bert (1989), which takes into account the influence of bending and shear in the facings as well as in the core, and results in constitutive equations of the same type as eqns (1) and (2). These equations also hold approximately for shear-deformable theories of laminated beams and for latticed beams. Furthermore, one-dimensional specializations of sixth-order plate theories, see Reissner (1985), can be easily incorporated. For a recent refined theory for isotropic beams of circular cross-section, fitting to eqns (1) and (2) see Valisetty (1990).

It is not within the scope of this contribution to argue the accuracy of the improvements of the Bernoulli–Euler theory with respect to exact solutions of the theory of elasticity, or about the physical sense, correctness and consistency of some of these refinements. For these questions the reader is referred to the cited papers, and also to Nicholson and Symmonds (1977, with various discussions), Hutchinson (1981, 1987), Levinson (1987a) and Rychter (1987).

It is emphasized, however, that a comparison between the outcomes of various theories for different support and loading conditions leads to surprising differences as well as to coincidences, see Levinson (1978b), where the Timoshenko theory and Levinson's theory have been compared.

Consequently, it is necessary to classify and predict the results of the refined theories in a systematic manner. In the following, this is done by means of a consistent analogy to the Bernoulli–Euler theory. As a starting point, note from eqns (1) and (2) and Table 1 that all the referenced refined theories can be put into a unified form with respect to the generalized coordinates and forces, w, ψ, and M, Q, respectively, which is due to the introduction of the tracer β.

In order to introduce this analogy, a formal similarity between eqn (1) and the constitutive equation of a Bernoulli–Euler beam under the action of an imposed curvature loading κ is noted (in the following, a bar refers to the Bernoulli–Euler theory):

$$\bar{M} = -B(\bar{w}_{,xx}+\kappa). \tag{3}$$

This type of loading results from sources of self-stress or from imposed dislocation fields in the beam; a corresponding example is the case of an imposed thermal curvature, $\kappa = \alpha m_\theta$, where $m_\theta = (1/J)_A \int \theta z \, dA$ denotes the first cross-sectional moment of the temperature θ, and J is the cross-sectional moment of inertia [see Ziegler and Irschik (1987)]. A further structural example for self-stresses are assembly stresses, e.g. due to kinks in the axis of a redundant Bernoulli–Euler beam, corresponding to imposed singular dislocation-type sources.

Obviously, differences between the Bernoulli–Euler solution and the solution of a refined theory must be due to a fictitious self-stress-type loading, because the former already satisfies the equilibrium conditions with respect to the lateral force loading. Of course, the completeness of this correspondence has to be proven, especially with respect to the support conditions. Subsequently, this proof is achieved by means of the axiomatic principle of virtual work. Thereby κ is connected to the imposed lateral force loading p of the original problem:

$$\kappa = -\beta(w_{,x}+\psi)_{,x} = -\beta Q_{,x}/S = p\beta/S, \tag{4}$$

which is the result of comparing eqns (1) and (3), using the equilibrium condition $Q_{,x} = -p$ as well as eqn (2). Note from eqn (4) that the factor β/S, which has been introduced in Table 1 in dimensionless form, characterizes the solution of a specific refined theory.

A PROBLEM-ORIENTED FORMULATION OF THE PRINCIPLE OF VIRTUAL WORK

Consider the state of equilibrium of a laterally-loaded beam according to one of the refined theories mentioned above. The principle of virtual work is applied to this state, where the influence function (Green's function) $\bar{w}^*(\xi, x)$ according to the Bernoulli–Euler theory is used as a special virtual deformation (the superscript * refers to a single force in the following):

$$\int p(\xi)\bar{w}^*(\xi, x)\,\mathrm{d}\xi + \int M(\xi)\bar{w}^*_{,\xi\xi}(\xi, x)\,\mathrm{d}\xi = 0. \tag{5}$$

In eqn (5), $\bar{w}^*(\xi, x)$ is the deflection at the point ξ, assuming the beam to be rigid in shear, due to a (dummy) single unit force loading applied at x. The first integral gives the virtual work of the (original) external forces p, and the second that of the corresponding internal ones; this sum has to vanish, where the integrals have to be extended over the whole structure. Note that $\bar{w}^*$ is a kinematically-admissible deformation field, because it is small by definition, and there are no contradictions with respect to the support conditions.

The bending moments M do their elementary work at the virtual curvature $\bar{\psi}^*_{,\xi} = -\bar{w}^*_{,\xi\xi}$, see eqn (2) with $S \to \infty$, and of course there is no virtual work of the shearing force Q or any other stress resultant done at a deformation field obeying the assumptions for Bernoulli–Euler beams. In conclusion, the kinematical constraints of this classical theory do not violate the requirement of virtual admissibility with respect to one of the refined theories. Thus, eqn (5) is complete.

In a second step, the principle of virtual work is applied to the dummy problem, i.e. to the Bernoulli–Euler beam under the action of a single unit force in x, where the deflections w (due to the original loading p, according to a refined theory) are used as the virtual deflections. [For the principle of virtual work applied to beams using the Mohr–Maxwell dummy unit load technique see Chwalla and Parkus (1961, pp. 308–315).] The virtual work statement of this auxiliary problem becomes

$$1w(x) + \int \bar{M}^*(\xi, x)w_{,\xi\xi}(\xi)\,\mathrm{d}\xi + \sum_j \bar{M}^*(\xi_j, x)[w_{,x}(\xi_j)] = 0. \tag{6}$$

The first term in eqn (6) gives the virtual work of the unit dummy load. The second one is the work of the corresponding internal forces, where the virtual curvature $(-w_{,\xi\xi})$ has to be used, following the requirements of the Bernoulli–Euler theory. Within this theory, the influence of the shearing forces $\bar{Q}^*$ or any other type of stress resultant upon the deformation has to be neglected. Contrary to eqn (5), where a virtually-admissible deformation field has been chosen, the refined deflection w, however, generally does not satisfy the support conditions of the Bernoulli–Euler-type dummy problem. This is due to the fact that boundary or continuity conditions of a refined theory have to be formulated in the rotation ψ

rather than in the slope $w_{,x}$. In the case of (rigidly) clamped ends or for intermediate point supports, the (refined) slope $w_{,x}$ generally does not vanish, or it is not continuous, respectively. Thus, rotational constraints have to be released according to Lagrange's free-body concept for virtual admissibility of the refined solution. Accordingly, the last term in eqn (6) gives the virtual work of the released dummy bending moments, where

$$[w_{,x}(\xi_j)] = w_{,x}(\xi_j^+) - w_{,x}(\xi_j^-)$$

denotes the jump in slope at the jth support with removed rotational constraint. Noting, however, the continuity condition $[\psi(\xi_j)] = 0$, and using eqn (2), leads to

$$[w_{,x}(\xi_j)] = R_j/S, \tag{7}$$

where $R_j = [Q(\xi_j)]$ is the reaction force at this jth support. Furthermore, again according to the principle of virtual work, it can be shown that

$$\bar{M}^*(\xi_j, x) = -\bar{w}^\Delta(x, \xi_j), \tag{8}$$

where $\bar{w}^\Delta(x, \xi_j)$ denotes the deflection in x due to a unit jump in slope applied at ξ_j (according to the Bernoulli–Euler theory): $\Delta = [\bar{w}^\Delta_{,\xi}(\xi_j, \xi_j)] = 1$. Equation (8) is a specialization of the so-called kinematical method by Mohr and Land for influence functions, compare e.g. Grüning (1914, pp. 476–484). (It is noted that eqn (8) can be derived by a two-stage procedure similar to that given above: in a first step, the principle of virtual work is applied to the beam rigid in shear under the action of the dummy force, where the jth support is released and $\bar{w}^\Delta$ is used as the virtual deformation. In order to show that the corresponding virtual work of the internal forces vanishes, the equilibrium of the beam under the action of the imposed dislocation Δ is considered in a second step, taking $\bar{w}^*_{,\xi\xi}$ as the virtual curvatures.)

ANALOGY AND EXAMPLES

Using the constitutive eqns (1) and (3), those integrands in the virtual work statements, eqns (5) and (6), which correspond to the work of the internal forces, can be expressed in terms of static quantities only:

$$\bar{M}^* w_{,\xi\xi} = -\bar{M}^* M/B - \bar{M}^* \kappa, \tag{9}$$

$$M \bar{w}^*_{,\xi\xi} = -M\bar{M}^*/B, \tag{10}$$

with κ of eqn (4). Thus, eqn (6) can be inserted into eqn (5), which gives

$$w = \int p\bar{w}^* \,\mathrm{d}\xi + \int \kappa \bar{M}^* \,\mathrm{d}\xi + \sum_j (R_j/S)\bar{w}^\Delta(\xi_j), \tag{11}$$

where eqns (7) and (8) have been used.

Equation (11) forms a complete analogy between the Bernoulli–Euler theory and the refined theories, decomposing the solutions of the latter into three Bernoulli–Euler-type parts:

$$w = \bar{w}_1 + \bar{w}_2 + \bar{w}_3, \tag{12a}$$

where the same decomposition holds for M and Q:

$$M = \bar{M}_1 + \bar{M}_2 + \bar{M}_3, \tag{12b}$$

$$Q = \bar{Q}_1 + \bar{Q}_2 + \bar{Q}_3. \tag{12c}$$

Equation (12b) follows by inserting (12a) into (1), noting eqns (3) and (4). Equation (12c) is then due to the equilibrium condition $M_{,x} = Q$. Having calculated w and Q by analogy, the generalized cross-sectional rotation ψ follows from eqn (2). The mechanical meaning of this partitioning is the following:

The part $\bar{w}_1$ denotes the associated Bernoulli–Euler solution, due to the imposed lateral force loading p:

$$\bar{w}_1(x) = \int p(\xi)\bar{w}^*(x,\xi)\,d\xi, \tag{13}$$

where the reciprocity theorem of Maxwell, $\bar{w}^*(\xi,x) = \bar{w}^*(x,\xi)$ [e.g. Grüning (1912, pp. 491–494)] has been used in the first integral of eqn (1). (Again, this theorem can be considered as the result of a problem-oriented, two-stage application of the principle of virtual work, using the states of equilibrium due to single forces applied in x and ξ, respectively, and equating the virtual work of the internal forces.) Equation (13) simply reflects the principle of superposition in linear structures.

The second part $\bar{w}_2$ corresponds to the deflection of a Bernoulli–Euler beam loaded by the imposed curvature κ of eqn (4) (recall that κ is proportional to the original lateral loading p):

$$\bar{w}_2(x) = \int \kappa(\xi)\bar{M}^*(\xi,x)\,d\xi. \tag{14}$$

This expression corresponds to Maysel's integration method of thermoelasticity, see Ziegler and Irschik (1987), where a two-stage derivation (including virtually-inadmissible deformation fields and comparing the equilibrium states due to κ and a single dummy force) has been given. Equation (14) gives the correction of the classical solution with respect to the effect of the distributed lateral loading. Noting eqn (4), it is seen from Table 1 that this correction is the same in the theories of Timoshenko (1921) and of Levinson (1981). Equation (14) yields identical results for all three versions of the refined theory derived by Rehfield and Murthy (1982). The ratio between these two sets of corrections is:

$$\bar{w}_{2\mathrm{T,L}}/\bar{w}_{2\mathrm{RM}} = 8(1+\nu)/(8+5\nu) \tag{15}$$

where T, L and RM stand for the Timoshenko, Levinson and Rehfield–Murthy theory, respectively. Additionally, there is [R refers to the refined theory of Rychter (1988)]:

$$\bar{w}_{2\mathrm{T,L}}/\bar{w}_{2R} = 12(1+\nu)/(12+5\nu). \tag{16}$$

All of the solutions for $\bar{w}_2$ coincide in the case where $\nu = 0$.

Using the properties of the Dirac delta function, it is seen from eqn (14) that the refined theories result in a jump in slope due to a single force:

$$\bar{w}_2^*(x,\bar{\xi}) = \int \delta(\bar{\xi}-\xi)\bar{M}^*(\xi,x)\,d\xi = -\bar{w}^\Delta(x,\bar{\xi}), \tag{17}$$

where $\bar{\xi}$ denotes the point of application, and eqn (8) has been used. Consequently, eqn (14) will give a non-trivial contribution for higher-order load singularities, if it is assumed that they result from single forces perpendicular to the beam axis, see the results of Stamm and Witte (1974, p. 34) for couples applied to a Timoshenko beam.

Furthermore, it has been shown by Ziegler and Irschik (1987) that an integral of the type (14) vanishes for all x, if the distribution of κ is proportional to the corresponding bending moment distribution $\bar{M}_2 = -B(\bar{w}_{2,xx}+\kappa)$. Because $\bar{M}_2$ is of the eigenstress type, it is due to reaction forces in redundant beams, and therefore it is spanwise linearly distributed. Thus, the vanishing of the correction term $\bar{w}_2$ can easily be predicted.

Note, on the other hand, that $\bar{w}_2$ is the only correction term in the statical determinate case of a one-span, simply-supported beam, where no rotational boundary conditions are involved. For an example of the coincidence of the Timoshenko and the Levison theory in that case, see Levinson (1987b) for the uniformly-loaded beam. The validity of eqn (15) for this example can be proven by means of results given in Rehfield and Murthy (1982). The corresponding solution of eqn (14) itself is $\bar{w}_2 = \kappa(ax-x^2)/2$ [e.g. Ziegler and Irschik (1987)] which—using eqn (4) and Table 1—coincides with the correction terms given in the papers referenced above. Here, a denotes the span.

The third part $\bar{w}_3$ in eqn (14) corresponds to the deflection of the Bernoulli–Euler beam due to imposed singular dislocations of rotational type (kinks, jumps in slope):

$$\bar{w}_3(x) = \sum (R_j/S)\bar{w}^{\Delta}(x,\xi_j), \tag{18}$$

where the kinks have to be applied at those supports of the Bernoulli–Euler beam which correspond to rotational boundary or continuity conditions in the refined theories. At the jth support, the amount of kink is R_j/S. Equation (18) gives the correction of the Bernoulli Euler theory with respect to the support conditions.

Consider, for example, the simple case of a cantilever beam under the action of a tip force F. In this case, we have $\bar{w}_2 = 0$. The reaction force at the clamped end $x = 0$ is $R = F$, and therefore $\bar{w}_3(x) = Fx/S$. Using the proper value of S from Table 1, coincidence with the results presented in Levinson (1987b) and Rychter (1988) is achieved.

Consider, furthermore, the case of a clamped clamped beam, $0 \leq x \leq a$, with a uniformly-distributed lateral loading p. The reaction force and clamping moment at $x = 0$ according to the Bernoulli–Euler theory are $\bar{R}_{10} = pa/2$, $\bar{M}_{10} = pa^2/12$, respectively. From global equilibrium and symmetry considerations, we have $\bar{R}_{20} = \bar{R}_{30} = 0$, and therefore total $R_0 = \bar{R}_{10}$. The moment due to κ is $\bar{M}_{20} = -B(p\beta/S)$, and the kinks at the clamped ends give $\bar{M}_{30} = 2B(R_0/S)/a = B(p/S)$. The correction term $(\bar{M}_{20}+\bar{M}_{30})$ therefore vanishes in the Timoshenko theory with $\beta = 1$, see Table 1. This has been noted by Rehfield and Murthy (1982), who derived non-vanishing expressions for all three of their theories. Using the values for S and β/S listed in Table 1, coincidence with these results is achieved.

For more general cases of statical indeterminate beams the R_js in eqn (18) are not known in advance, but follow from $R_k = \bar{R}_{1k}+\bar{R}_{2k}+\bar{R}_{3k}$, where $\bar{R}_{3k}$ is a linear function of all kinks R_j/S. As a simple example of that type, consider a clamped–hinged beam of span l with uniformly-distributed loading p. At the clamped end, we have $\bar{R}_{10} = 5pl/8$, $\bar{R}_{20} = 3(\beta p/S)B/2l$ and $\bar{R}_{30} = -3(R_0/S)B/l^2$. By adding and equating for R_0 in the case $\beta = 1$, coincidence with a result by Stamm and Witte (1974, p. 38) for a Timoshenko-type theory is achieved.

All the results for the Bernoulli–Euler theory applied to solve the preceding examples can be taken from standard textbooks on structural mechanics, see for example, the tables presented in Duddeck and Ahrens (1982, pp. 602–625).

Finally, with respect to a widely-used procedure for the Timoshenko theory it is noted that only in the simple case of a one-span, hinged–hinged beam do we have $\bar{w}_1 = w_1$ and $\bar{w}_2 = w_2$, with

$$w_{1,xx} = -M/B, \quad w_{2,xx} = -p/S, \quad w = w_1+w_2. \tag{19}$$

w_1, w_2 denote the so-called bending and shear part of the deflection [see von Karman (1910, p. 334)]. In the statical determinate case of a cantilever, it is possible to set $w_2 = \bar{w}_2+\bar{w}_3$. In the general case of redundant beams, however, w_1 and w_2 are coupled by means of the boundary conditions, and there is no analogy between this bending and shear parts and solutions of the Bernoulli–Euler theory.

CONCLUSION

Using the principle of virtual work in a special, problem-oriented formulation, an analogy has been established between some refined theories of the bending of beams and the classical Bernoulli–Euler theory. Corrections to the classical solution have been interpreted as being due to additional sources of self-stress acting in the Bernoulli–Euler beam. Two types of self-stress loadings are involved: distributed curvature loadings, similar to thermal curvatures, and singular dislocations of kink type, the latter accounting for the change in rotational support constraints.

Clamped and hinged boundary conditions as well as intermediate point supports have been considered. For comparison's sake, the examples have been restricted to beams of constant, rectangular cross-section. (Extensions to other geometries are self-evident.) By means of this analogy, refined expressions for deflections, bending moments and shear forces can be evaluated from standard, well-known Bernoulli–Euler-type solution procedures of structural mechanics. The two-dimensional fields of deformations, stresses and strains are calculated afterwards according to the specific rules of the refined theories.

At first glance, the strategy of establishing analogies between theories for one and the same problem seems to be surprising. The background of this procedure with respect to the Theory of Science, therefore, is discussed in the Appendix, where some further applications are mentioned.

REFERENCES

Chwalla, E. and Parkus, H. (1961). Strength of materials. In *Civil Engineering References Book* (Edited by J. Crombie), 2nd Edn. Butterworth, London.

Duddeck, H. and Ahrens, H. (1982). Statik der Stabtragwerke. In Betonkalender (Edited by G. Franz), Vol. I. W. Ernst & Sohn, Berlin.

Fotiu, P., Irschik, H. and Ziegler, F. (1989). Dynamic plasticity: structural drift and modal projection. In *Nonlinear Dynamic Engineering Systems* (*Proc. IUTAM Symp.*, Stuttgart, 1989) (Edited by W. Schiehlen). Springer, Berlin.

Gordaninejad, F. and Bert, C. W. (1989). New theory for bending of thick sandwich beams. *Int. J. Mech. Sci.* **31**, 925–934.

Grüning, M. (1914). Theorie der Baukonstruktionen I. In *Encyklopaedie der mathematischen Wissenschaften* (Edited by F. Klein and C. Müller), Vol. 4, Part 4.4, p. 494. B. G. Teubner, Leipzig.

Hutchinson, J. R. (1981). Transverse vibrations of beams, exact versus approximate solutions. *J. Appl. Mech.* **48**, 923–928.

Hutchinson, J. R. (1987). A comparison of Mindlin and Levinson plate theories. *Mech. Res. Comm.* **14**, 165–170.

Irschik, H. (1982). Eine Analogie zwischen Lösungen für schubstarre und schubelastische Platten. *ZAMM* **62**, T129–T131.

Irschik, H. (1985). Membrane-type eigenmotions of Mindlin plates. *Acta Mech.* **55**, 1–20.

Irschik, H., Heuer, R. and Ziegler, F. (1989). Dynamic analysis of polygonal Mindlin plates on two-parameter foundations using classical plate theory and an advanced BEM. *Comput. Mech.* **4**, 293–300.

Irschik, H. and Ziegler, F. (1988). Dynamics of linear elastic structures with selfstress: a unified treatment for linear and nonlinear problems. *ZAMM* **68**, 199–205.

Irschik, H. and Ziegler, F. (1989). Evolutionary damage prediction of discrete and continuous elastoplastic structures with random earthquake excitation. In *Computational Mechanics of Reliability Analysis* (Edited by T. Belytechko and W. K. Liu). Elmepress, Lausanne.

Levinson, M. (1981). A new rectangular beam theory. *J. Sound Vib.* **74**, 81–87.

Levinson, M. (1987a). On higher order beam and plate theories. *Mech. Res. Comm.* **14**, 421–424.

Levinson, M. (1987b). Consistent and inconsistent higher order beam and plate theories: some surprising comparisons. In *Refined Dynamical Theories of Beams, Plates and Shells and their Applications* (Edited by I. Elishakoff and H. Irretier). Springer, Berlin.

Mach, E. (1883). *Die Mechanik in ihrer Entwicklung historisch kritisch dargestellt.* F. A. Brockhaus, Leipzig.

Mach, E. (1902). Die Ähnlichkeit und die Analogie als Leitmotiv der Forschung. *Annalen der Naturphilosophie* 5–14.

Mura, T. (1987). *Micromechanics of Solids*, 2nd Edn. Martinus Nijhoff, The Hague.

Nicholson, J. W. and Simmonds, J. G. (1977). Timoshenko beam theory is not always more accurate than elementary beam theory. *J. Appl. Mech.* **44**, 337–338.

Plantema, F. J. (1966). *Sandwich Construction.* John Wiley, New York.

Popper, R. K. (1966). *Of Clouds and Clocks.* Washington University Press, St Louis, Miss.

Popper, R. K. (1970). A realist view of logic, physics and history. In *Physics, Logic and History* (Edited by W. Yourgau and A. D. Breck). Plenum Press, New York.

Rehfield, L. W. and Murthy, L. N. (1982). Towards a new engineering theory of bending: fundamentals. *AIAA Jl*. **20**, 693–699.

Reissner, E. (1985). Reflections on the theory of elastic plates. *Appl. Mech. Rev.* **38**, 1453–1464.

Reißner, H. (1931). Eigenspannungen und Eigenspannungsquellen. *ZAMM* **11**, 1–8.

Rychter, Z. (1987). On the shear coefficient in beam bending. *Mech. Res. Comm.* **14**, 379–385.

Rychter, Z. (1988). An engineering theory for beam bending. *Ing.-Arch.* **58**, 25–34.

Stamm, K. and Witte, H. (1974). *Sandwichkonstruktionen*. Springer, New York.

Timoshenko, S. P. (1921). On the correction for shear of the differential equation for transverse vibrations of prismatic bars. *Phil. Mag. Ser.* 6, **41**, 744–746.

von Karman, T. (1910). Festigkeitsprobleme in Maschinenbau. In *Encyklopaedie der mathematischen Wissenschaften* (Edited by F. Klein and C. Mueller), Vol. 4, Part 4.3. B. G. Teubner, Leipzig.

Valisetty, R. R. (1990). Refined bending theory for beams of circular cross section. *J. Engng Mech.* **116**, 2072–2079.

Voss, A. (1901). Die Prinzipien der rationellen Mechanik. In *Encyklopaedie der mathematischen Wissenschaften* (Edited by F. Klein and C. Mueller), Vol. 4, Part 1.1. B. G. Teubner, Leipzig.

Ziegler, F. and Irschik, H. (1985). Dynamic analysis of elastic–plastic beams by means of thermoelastic solutions. *Int. J. Solids Structures* **21**, 819–829.

Ziegler, F. and Irschik, H. (1987). Thermal stress analysis based on Maysel's formula. In *Thermal Stresses II* (Edited by R. B. Hetnarski). Elsevier, Amsterdam.

APPENDIX

Following K. R. Popper, the situation of competitive theories, which is characteristic for problems in natural sciences, results from the critical discussion of a basic, classical theory, where some limitations or shortcomings lead to the development of refined theories [e.g. Popper (1966, 1970)]. (These refined theories have to be critically examined themselves, giving rise to a re-start of the process.) The refined or extended theories, generally carrying a higher content of information, are superior to the classical one, and the latter should be omitted from a scientific point of view.

In practice, however, this clearance seldom takes place, but refined theories do not win the competition for acceptance during a longer period of time. The main reasons for this are a lack of continuing education and a certain moment of inertia in developing methods for the convenient treatment of refined theories, while methods for the classical theory remain a field of interest.

It seems reasonable to overcome these drawbacks by applying the principle of analogy to Popper's scheme of scientific development. In the field of mechanics, this principle was proposed by E. Mach [see Mach (1883, 1902) and Voss (1901, p. 20)]. Commonly, it is used by transferring well-known methods from one problem to an entirely different one.

If it is possible, however, to establish an analogy between the refined theories and the classical one itself, problems can easily be solved according to the rules of refined theories using widely-known methods for the classical one. Hence, the practical acceptance of the refined theories may thereby be increased, while their results can be classified in a systematic manner by considering them in the light of the classical theory. Paralleling the terminology of Popper (1966), the classical, elementary theory thus finds an ecological niche.

Of course, in order to establish such an analogy, the range of applicability of the elementary theory has to be extended slightly, but without leaving its basic limitations.

Above, in the context of beam theories, this has been done by considering sources of self-stress in the classical formulation, in addition to the (original) lateral loading. This type of source–loading has been extensively studied [see Reißner (1931) and Mura (1987)]. For structural applications of thermal loadings in Bernoulli–Euler beams, see Ziegler and Irschik (1987).

In a similar manner, an analogy between the elementary theory of the bending of plates and their shear deformable extensions has been established earlier for simply-supported plates of polygonal planform [Irschik (1982); Irschik (1985); Irschik *et al.* (1989)].

Furthermore, problems of the vibrations of structures, which are driven into the inelastic range by severe loadings, have been treated by considering the non-linear part of strain to be analogous to additional sources of self-stress in the linear elastic "background" structure. Thus, powerful methods of classical linear elastic dynamics, such as influence functions or modal analysis, become applicable in the extended situation of physical non-linearity by analogy [see Ziegler and Irschik (1985)]. For a review, see Irschik and Ziegler (1988). For the application to elasto-viscoplastic structures with material degradation, see Fotiu *et al.* (1989), and for random vibrations of elastoplastic structures see Irschik and Ziegler (1989).

Likewise to the analogy for refined beam theories, these applications serve as examples for the use of well-known classical methods of analysis in the context of advanced, "higher order" or extended theories.